WEEK 11:

Last time: $\mathrm{Spa}(A(\frac{f_1,\dots,f_n}{g}), A(\frac{f_1,\dots,f_n}{g})^+) \overset{\text{homeo}}{\cong} R(\frac{f_1,\dots,f_n}{g})$.

modulo: $\mathrm{Spa}(A,A^+)$ q.cpt.

instead we showed $\mathrm{Spv}(k)$ q-cpt.

$\Rightarrow$ $\mathrm{Spv}(A)$ q-cpt (spectral)

$\mathrm{Spv}(A,I)$ q-cpt (spectral).

Definition: X is quasi-separated if and only if diagonal is quasi-compact.

[$\Rightarrow$ $Y,Z \subseteq X$ quasi-compact then $Y \cap Z$ q-cpt.]

X q-cpt, q-sep, $Y \subseteq X$ constructible if $Y = U \cap (X \setminus V)$ where U, V are open q-cpt.

Homework 12: Does this agree with scheme theoretic notion of constructible.

$Y \subseteq X$ pro-constructible if Y is an intersection of constructible sets.

Remark: Clearly $\mathrm{Spa}(A,A^+) = \bigcap_{a \in A^+} \mathrm{Spa}(A, \{a\}) = \bigcap_{a \in A^+} \underset{\text{can show that q.cpt.}}{R(\frac{1,a}{1})}$ from this we get that $\mathrm{Spa}(A^+,A)$ is pro-const.

Argument. $\mathrm{Spv}(A)$ q.cpt $\longrightarrow$ $\mathrm{Spv}(A,I)$ q.cpt $\longrightarrow$ $\mathrm{Spv}(A,I) \overset{\text{closed}}{\supseteq} \mathrm{Cont}(A)$ q.cpt

$\downarrow$

$\mathrm{Spa}(A,A^+) \subseteq \mathrm{Cont}(A)$ q.cpt.

Definition: X spectral if

(i) q-cpt, q-spt (ii) $\exists$ base of q-cpt open (iii) sober (every irr. closed has a unique generic pt).

Homework 13*: example of $\mathrm{Spv}(A) \supseteq \mathrm{Cont}(A)$ not closed.

Theorem: X spectral, $Y \subseteq X$, TFAE

Y is pro-constructible $\Leftrightarrow$ Y spectral & $Y \subseteq X$ spectral.

We also proved last time:

Lemma: A top. ring $A_0 \subseteq A$ subring TFAE

(i) A f-adic & A_0 a ring of def.

(ii) $A_0 \subseteq A$ open & bounded and $\exists\, U \subseteq A$ additive subgp. & $\exists\, T \subseteq U$ finite st.

(a) $\{U^n\}$ fund. system of neigh. of 0.

(b) $T.U = U^2 \subseteq U$

Corollary: We can take $A_0 = A^\circ \Leftrightarrow A^\circ$ is bounded. (by definition A° bounded means uniform).

Corollary: A f-adic (A_0, I) & (A_0', I') couples of definition.

$\Rightarrow$ $A_0.A_0'$ & $A_0 \cap A_0'$ are rings of definition.

Repairing the definition of adic-morphisms.

$f: A \to B$ A,B f-adic. Then we said that f is adic if there exists (A_0, I) & (B_0, J) couples of definition such that $f(A_0) \subseteq B_0$ and $f(I).B$ ideal of definition.

Homework 14: $f: A \to B$ adic-morphism. A_0, B_0 rings of def such that $f(A_0) \subseteq B_0$ and $I \subseteq A_0$ ideal of definition then show that $f(I).B_0$ an ideal of def.

Homework 15* (experimental): $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$. Recall $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is called adeles. Show that $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is not f-adic.

Homework 16: in $\mathbb{Q}_p\langle x\rangle$ what is the intersection $R\left(\frac{x,p}{p}\right) \cap R\left(\frac{x,1,p}{p}\right)$? (think about non classical points).

Proposition: (A,A^+) affinoid, $\varphi: (A,A^+) \to (\hat{A}, \hat{A}^+)$ (adic morphism) int. closure of $A^+.\hat{A}$ in $\hat{A}$.
$\Rightarrow$ $\mathrm{Spa}(\varphi)$ is a homeomorphism & preserves rat. open sets.

Proof: ① $A \to \hat{A}$ is dense $\Rightarrow$ $\mathrm{Spa}(\varphi)$ is inj.
② $|\cdot| \in \mathrm{cont}(A) \Rightarrow$ extends uniquely to $\hat{A} \Rightarrow \mathrm{Spa}(\varphi)$ surj.
③ $\mathrm{Spa}(\varphi)^{-1}(\text{rational}) = \text{rational}$.
Remains to show $\mathrm{Spa}(\varphi)(\text{rational}) = \text{rational}$. Need to perturb rational open sets.

Proposition: A complete f-adic ring. $f_1,\dots,f_n, g \in A$ st. $(f_1,\dots,f_n) \subseteq_{\text{open}} A$.
$\Rightarrow \exists U \ni 0$ open st $\forall f_i' \in f_i + U$ $\forall g' \in g+U \Rightarrow (f_1',\dots,f_n') \subseteq_{\text{open}} A$ &

$$\mathrm{Cont}\left(\frac{f_1,\dots,f_n}{g}\right) = \mathrm{Cont}\left(\frac{f_1',\dots,f_n'}{g'}\right)$$

follows from the last perturbation lemma.

The hard part is to show that $(f_1',\dots,f_n')$ is open.

Lemma: A complete f-adic w/ couple of def. (A_0, I) Choose $I = (a_1,\dots,a_n)$ then $\forall a_i' \in a_i + I^2$, $(a_1',\dots,a_n') = I$.

Proof: $\psi: A_0^n \longrightarrow I$ morphism of A_0-mods. Want to show ψ is surjective.
$(b_1,\dots,b_n) \mapsto \sum b_i a_i'$

Note that A_0^n & I are complete A_0-modules. Then as $\psi(I^s A_0^n) \subseteq I^{s+1} A_0^n$ suffices to show that $\mathrm{Gr}_I \psi$ is surjective. But that is trivial,

$$I^s A_0^n / I^{s+1} A_0^n \xrightarrow[\cdot a_i = \cdot a_i']{} I^{s+1}/I^{s+2}$$

is easily surj.

Proof of the proposition: Need to find U st $Ua_{ij} \subseteq I^2$ $\forall i,j$ where $Ua_{ij} \subseteq I^2$. □

All this together shows that in $\mathrm{Spa}(A,A^+)$

$$R\left(\frac{f_1,\dots,f_n}{g}\right) \overset{\text{homeo}}{\cong} \mathrm{Spa}\left(A\left\langle\frac{f_1,\dots,f_n}{g}\right\rangle, A\left\langle\frac{f_1,\dots,f_n}{g}\right\rangle^+\right)$$

Final step in the construction of adic spaces

(A,A^+) affinoid, $T \subseteq A$ finite st $T.A$ open, $s \in A$ and let $U = R(T/s)$.
$\varphi: (A,A^+) \xrightarrow{\text{cont}} (B,B^+)$ morph. of affinoid rings st $\mathrm{im}(\mathrm{Spa}(\varphi)) \subseteq U$ with B complete.
i.e $\varphi(A^+) \subseteq B^+$

Then there is a unique topo. ring homo. $A\langle T/s\rangle \xrightarrow{\psi} B$ st

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[dr] & A\langle T/s\rangle \arrow[d] \\ & B \end{tikzcd}$$

commutes and $\psi(A\langle T/s\rangle^+) \subseteq B^+$.
(Δ)

Proof: By the univ. property of the localization, we need
① $\varphi(s) \in B^\times$ ② $\{\varphi(t/s) \mid t \in T\}$ is power-bounded.
We know $|\cdot| \in \mathrm{Spa}(B,B^+) \Rightarrow |\varphi(T)| \le |\varphi(s)| \neq 0$ $\forall t \in T$ if ① is known then $\left|\frac{\varphi(t)}{\varphi(s)}\right| < 1$. By some proposition from 3-4 weeks ago $\varphi(t)/\varphi(s) \in B^+$
then (Δ) and ② are automatic. Hence we only need to show $\varphi(s) \in B^\times$.

Theorem: $a \in (A,A^+)$ complete affinoid then $a \in A^\times \Leftrightarrow |a| \neq 0$ $\forall |\cdot| \in \mathrm{Spa}(A,A^+)$.